Spectral sequences for multidimensional persistence

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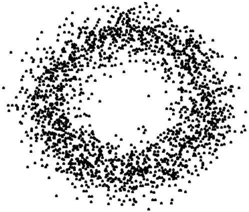
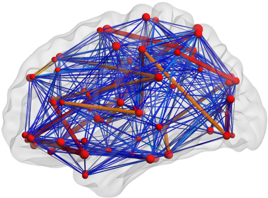
Topological data analysis

Applications: image analysis, shape study, sensor networks, viral evolution and propagation of epidemics, internet and social networks, biological networks . . .





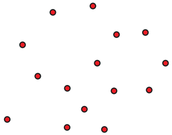
Topological methods:

- Transform data into global topological object (**simplicial complexes**)
- Study these objects through algebraic topology (**persistent homology**)

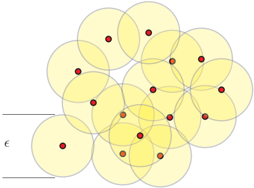
Plan of the talk

1. Persistent homology
   - data $\rightarrow$ simplicial complexes $\rightarrow$ filtrations $\rightarrow$ topological invariants
2. Spectral sequences vs Persistent homology
   - "same topological information"
3. Multidimensional case

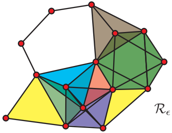
From data to simplicial complexes



(a) Point cloud



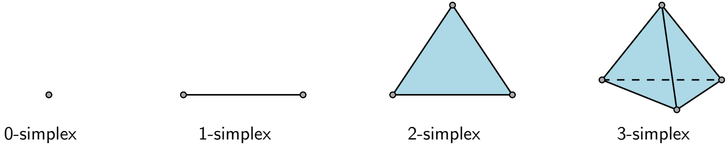
(b) Connections between points



(c) Simplicial complex (Rips)

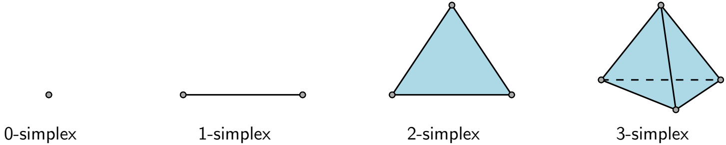
Simplicial complexes

Simplices:

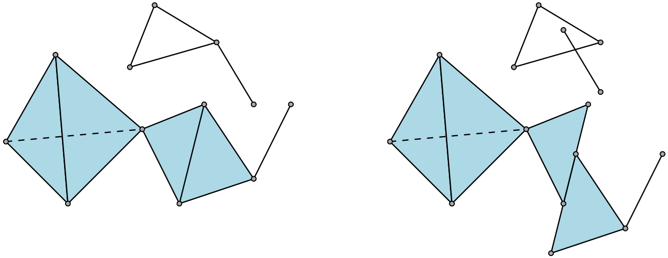


Simplicial complexes

Simplices:

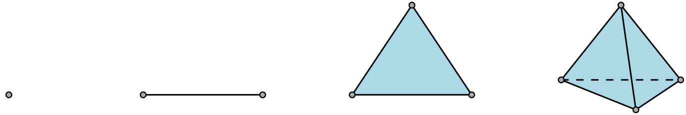


Simplicial complexes:



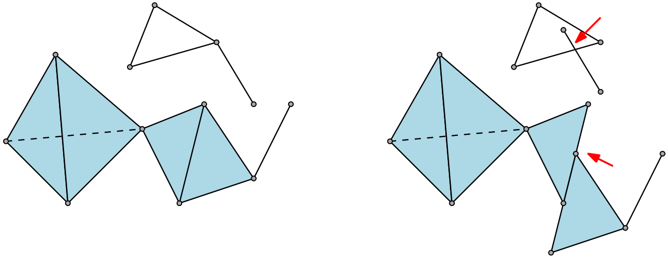
Simplicial complexes

Simplices:



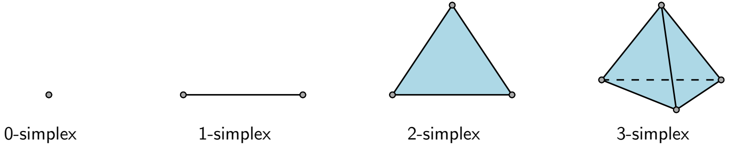
0-simplex 1-simplex 2-simplex 3-simplex

Simplicial complexes:

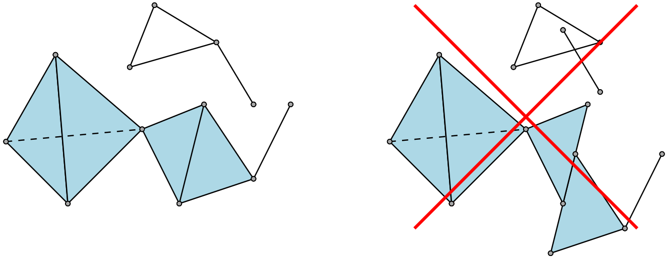


Simplicial complexes

Simplices:

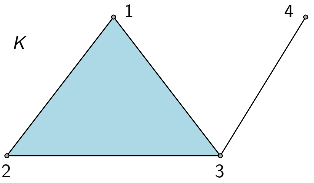


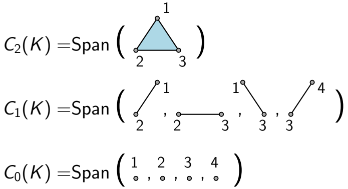
Simplicial complexes:



Homology: Chain complex of a simplicial complex

$C_n(K) := \{$linear combinations of n-simplices of $K\}$





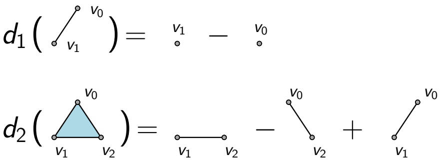
Chain complex of K:

$$C_\bullet(K): \quad \cdots \xrightarrow{d_3} C_2(K) \xrightarrow{d_2} C_1(K) \xrightarrow{d_1} C_0(K) \to 0$$

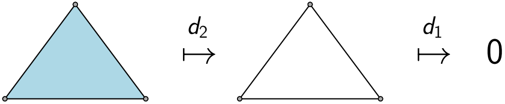
The **boundary operator**

$$d_n : C_n(K) \to C_{n-1}(K)$$

is the linear map defined on n-simplices as the alternating sum of $(n-1)$-faces:



Fundamental property: $d_n d_{n+1} = 0$



Homology groups and Betti numbers

Consider the chain complex of a simplicial complex K

$$C_\bullet(K): \quad \cdots \to C_{n+1}(K) \xrightarrow{d_{n+1}} C_n(K) \xrightarrow{d_n} C_{n-1}(K) \to \cdots$$

The fundamental property $d_n d_{n+1} = 0$ implies

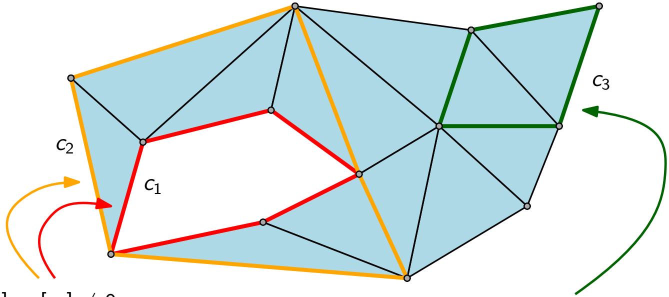
$$\operatorname{Im} d_{n+1} \subseteq \ker d_n \;\; (\subseteq C_n(K)).$$

The n-th **homology group** of K is defined as

$$H_n(K) := \frac{\ker d_n}{\operatorname{Im} d_{n+1}}$$

and its rank β_n is called n-th **Betti number**.

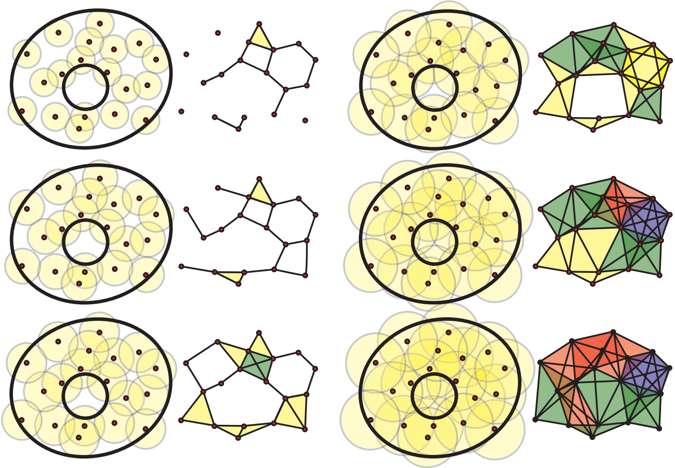
Homology classes



$[c_1] = [c_2] \neq 0$
in $H_1(K)$

$[c_3] = 0$ in $H_1(K)$

Filtration of simplicial complexes



Persistent homology

Given a filtration of simplicial complexes

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K,$$

for every $i \leq j$ the inclusion $K_i \subseteq K_j$ induces maps in homology

$$f_n^{i,j} : H_n(K_i) \to H_n(K_j).$$

For every $i \leq j$, we call the groups

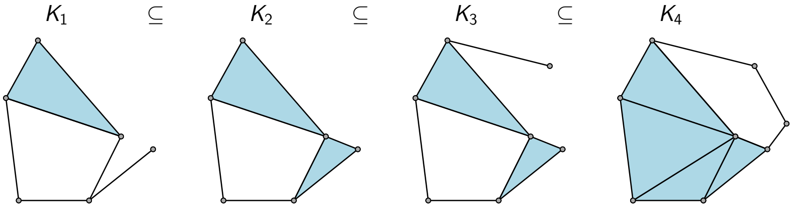
$$H_n^{i,j} := \operatorname{Im}(f_n^{i,j})$$

persistent homology groups. We define

$$\beta_n^{i,j} := \dim_k H_n^{i,j}$$

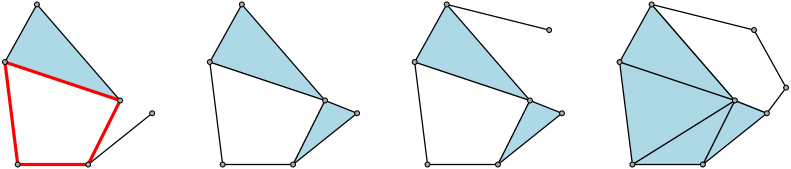
persistent Betti numbers.

Persistent homology



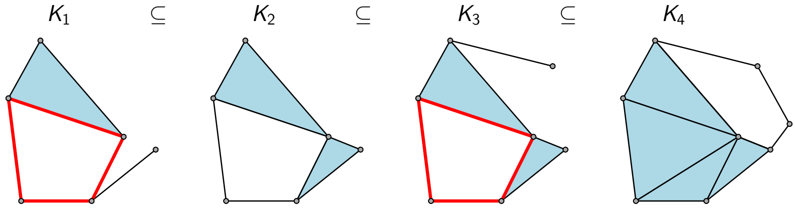
Persistent homology

$K_1 \quad \subseteq \quad K_2 \quad \subseteq \quad K_3 \quad \subseteq \quad K_4$



$[z] \in H_1(K_1)$

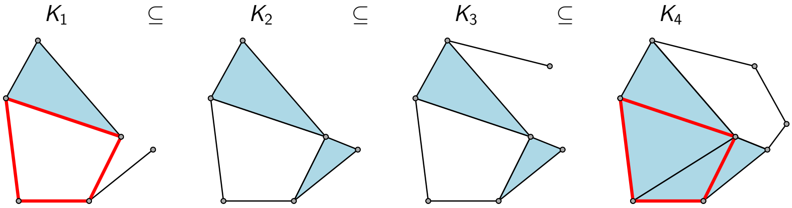
Persistent homology



$[z] \in H_1(K_1)$ $\qquad$ $[z] \in H_1(K_3)$

Persistent Betti numbers: $\beta_1^{1,3} = 1$

Persistent homology

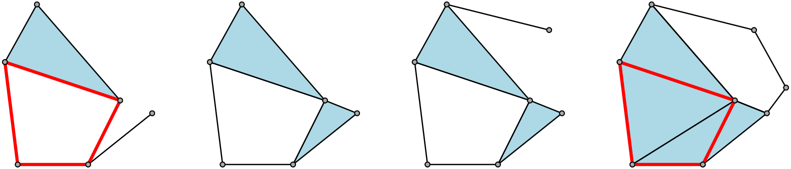


$[z] \in H_1(K_1)$ $\qquad$ $[z] = 0 \in H_1(K_4)$

Persistent Betti numbers: $\beta_1^{1,3} = 1$, $\beta_1^{1,4} = 0$

Persistent homology

$K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4$

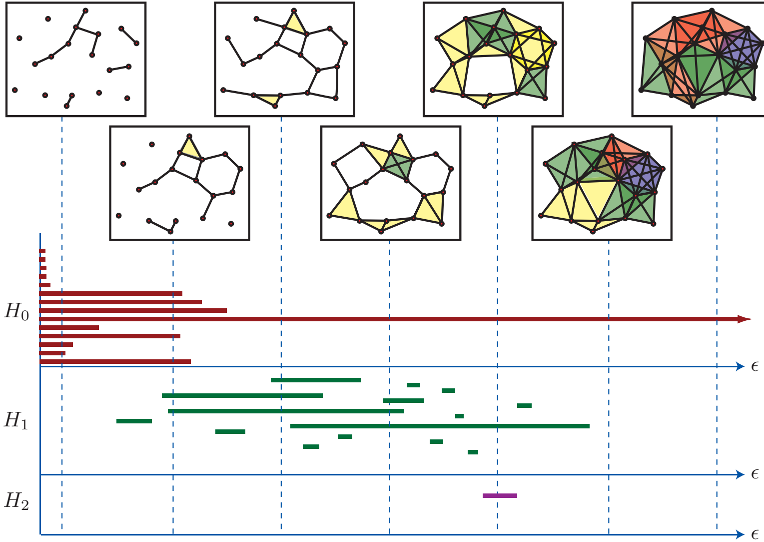


$[z] \in H_1(K_1)$ $\qquad$ $[z] = 0 \in H_1(K_4)$

Persistent Betti numbers: $\beta_1^{1,3} = 1$, $\beta_1^{1,4} = 0$, ...

Barcode for H_1 (counting "holes"):

Barcodes



Plan of the talk

1. Persistent homology
   - data → simplicial complexes → filtrations → topological invariants
2. Spectral sequences vs Persistent homology
   - "same topological information"
3. Multidimensional case

Spectral sequences

Filtration of a simplicial complex:

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq \; K_{p-1} \subseteq \; K_p \subseteq \cdots \subseteq \; K_m = K.$$

Filtration of a chain complex:

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{p-1} \hookrightarrow F_p \hookrightarrow \ldots \hookrightarrow F_m = C_\bullet(K),$$

where we denote $F_p := C_\bullet(K_p)$.

Leray ($\sim$ 1940): spectral sequences as a device for computing the homology $H_n(K)$ using the information contained in the filtration.

Spectral sequences

Filtration of a chain complex:

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{p-1} \hookrightarrow F_p \hookrightarrow \ldots \hookrightarrow F_m = C_\bullet(K)$$

Spectral sequence:

$$\{E_p^r\}_{p,r\in\mathbb{Z}}, \quad r \text{ ``page index''}, p \text{ ``filtration index''},$$

where

$$E_p^1 := H(F_p/F_{p-1})$$

and there exist differentials

$$\cdots E_{p+r}^r \xrightarrow{d} E_p^r \xrightarrow{d'} E_{p-r}^r \cdots \quad \text{with} \quad E_p^{r+1} \cong \ker d' / \operatorname{Im} d.$$

Convergence

There exists a filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = H(K)$ such that, for $r \geq r_0$,

$$E_p^r \cong V_p/V_{p-1}.$$

Spectral sequences

Filtration of a chain complex:

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{p-1} \hookrightarrow F_p \hookrightarrow \ldots \hookrightarrow F_m = C_\bullet(K)$$

Spectral sequence:

$$\{E^r_p\}_{p,r\in\mathbb{Z}}, \quad r \text{ ``page index''}, p \text{ ``filtration index''},$$

where

$$E^1_p := H(F_p/F_{p-1})$$

and there exist differentials

$$\cdots E^r_{p+r} \xrightarrow{d} E^r_p \xrightarrow{d'} E^r_{p-r} \cdots \qquad \text{with} \quad E^{r+1}_p \cong \ker d' / \operatorname{Im} d.$$

Convergence

There exists a filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = H(K)$ such that, for $r \geq r_0$,

$$E^r_p \cong V_p/V_{p-1}.$$

Spectral sequences vs Persistent homology

Given a filtration

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{p-1} \hookrightarrow F_p \hookrightarrow \ldots \hookrightarrow F_m = C_\bullet(K)$$

study the relationship between the spectral sequence $\{E^r_p\}$ and the persistent homology groups $\{H^{i,j} = \mathrm{Im}(H(F_i) \to H(F_j))\}$:
there are long exact sequences

$$\ldots H^{p-1,p+r-2} \to H^{p,p+r-1} \to E^r_p \to H^{p-r,p-1} \to H^{p-r+1,p} \ldots$$

The spectral sequence and the persistent homology groups "contain the same topological information" :

$$\dim_k E^r_p \quad \leftrightsquigarrow \quad \beta^{i,j} := \dim_k H^{i,j}$$

Plan of the talk

1. Persistent homology
   - data $\rightarrow$ simplicial complexes $\rightarrow$ filtrations $\rightarrow$ topological invariants
2. Spectral sequences vs Persistent homology
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Multidimensional persistent homology

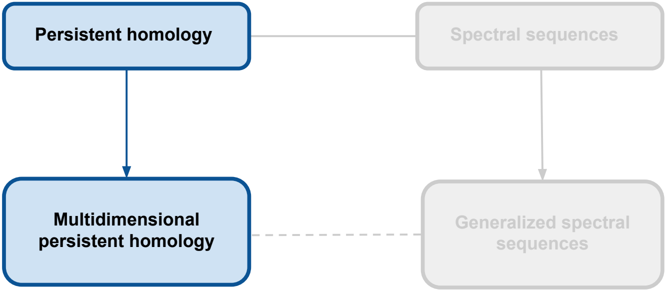
Multidimensional filtration of a simplicial complex:

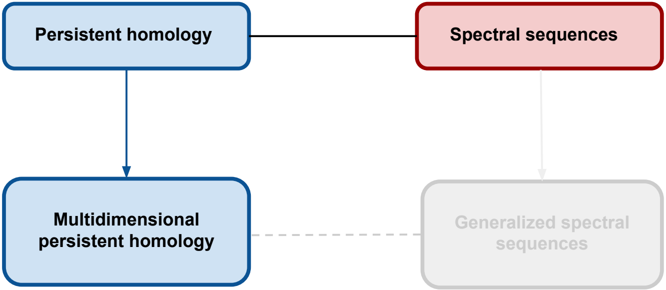
$$
\begin{array}{ccccccc}
K_{m1} & \hookrightarrow & K_{m2} & \hookrightarrow & \cdots & \hookrightarrow & K_{mm'} \\
\uparrow & & \uparrow & & & & \uparrow \\
\cdots & & \cdots & & & & \cdots \\
\uparrow & & \uparrow & & & & \uparrow \\
K_{21} & \hookrightarrow & K_{22} & \hookrightarrow & \cdots & \hookrightarrow & K_{2m'} \\
\uparrow & & \uparrow & & & & \uparrow \\
K_{11} & \hookrightarrow & K_{12} & \hookrightarrow & \cdots & \hookrightarrow & K_{1m'}
\end{array}
$$

Associated invariant: **rank invariant**

$$\beta_n^{P,Q} := \dim_k \operatorname{Im}(H_n(K_P) \to H_n(K_Q)), \quad P, Q \in \mathbb{Z}^2, P \preceq Q.$$

Unlike the 1-dimensional case (persistent Betti numbers), it is *not* a complete invariant.


Persistent homology
Spectral sequences
Multidimensional persistent homology
Generalized spectral sequences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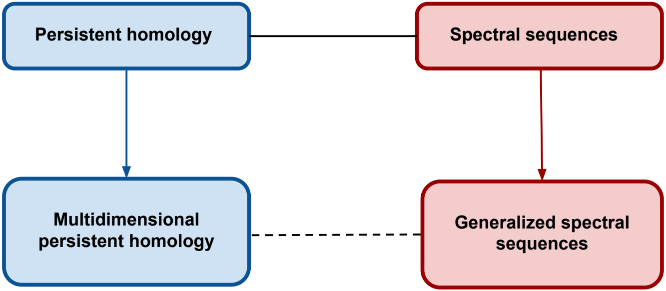
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Generalized spectral sequences

For a filtration $\{F_p\}_{p\in\mathbb{Z}}$ we have

$$E_p^1 = H(F_p/F_{p-1}) = \frac{\text{rel. cycles}}{\text{rel. boundaries}} = \frac{F_p \cap d^{-1}(F_{p-1})}{d(F_p) + F_{p-1}}$$

$$E_p^r = \frac{F_p \cap d^{-1}(F_{p-r})}{d(F_{p+r-1}) + F_{p-1}}$$

This can be generalized for a **filtration indexed over a poset** I, i.e. a collection of chain complexes $\{F_i\}_{i\in I}$ with $F_i \hookrightarrow F_j \iff i \leq j$, in the following way:

$$S[z,q,p,b] := \frac{F_p \cap d^{-1}(F_z)}{d(F_b) + F_q},$$

for all $z \leq q \leq p \leq b$ in I.

Generalized spectral sequences

For a filtration $\{F_p\}_{p\in\mathbb{Z}}$ we have

$$E_p^1 = H(F_p/F_{p-1}) = \frac{\text{rel. cycles}}{\text{rel. boundaries}} = \frac{F_p \cap d^{-1}(F_{p-1})}{d(F_p) + F_{p-1}}$$

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for all $z \leq q \leq p \leq b$ in I.

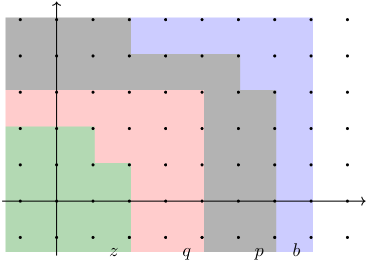
Spectral sequences for multidim. persistent homology

Consider the poset $I := D(\mathbb{Z}^n)$ of **downsets** of $\mathbb{Z}^n$, ordered wrt inclusion $\subseteq$. (A downset of $\mathbb{Z}^n$ is a subset $A \subseteq \mathbb{Z}^n$ such that if $P \in A$ and $Q \preceq P$, then $Q \in A$).

The terms of the generalized spectral sequence are of the form

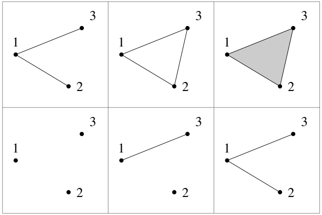
$$S[z, q, p, b] := \frac{F_p \cap d^{-1}(F_z)}{d(F_b) + F_q},$$

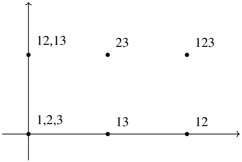
for all $z \subseteq q \subseteq p \subseteq b$ in $I = D(\mathbb{Z}^n)$.



Spectral sequences for multidim. persistent homology

How to define a filtration of chain complexes $\{F_p\}_{p \in I = D(\mathbb{Z}^n)}$ from a multidimensional filtration of simplicial complexes $\{K_P\}_{P \in \mathbb{Z}^n}$.





For every point $P \in \mathbb{Z}^n$ consider the simplices entering the filtration at K_P. Then, for $p \in I = D(\mathbb{Z}^n)$, set

$$F_p := \text{Span}\{\text{simplices entering the filtration at } K_P, \text{ for all } P \in p\}$$

Spectral sequences for multidim. persistent homology

Theorem

Under suitable hypotheses on $z_1, q_1, \dots, b_3 \in I = D(\mathbb{Z}^n)$, there exist differentials

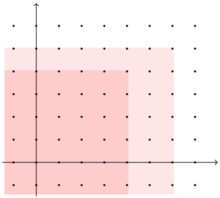
$$S[z_3, q_3, p_3, b_3] \xrightarrow{d} S[z_2, q_2, p_2, b_2] \xrightarrow{d'} S[z_1, q_1, p_1, b_1]$$

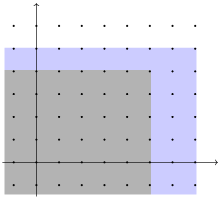
such that

$$\frac{\ker d'}{\operatorname{Im} d} \cong S[q_1, q_2, p_2, p_3].$$

Our idea: consider $\{\dim_k S[z, q, p, b]\}$ as invariant for the multidimensional filtration $\{K_P\}_{P \in \mathbb{Z}^n}$.

Spectral sequences for multidim. persistent homology

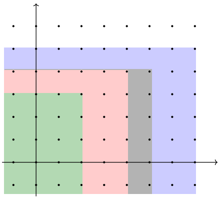




$S[-\infty, -\infty, q, q']$ $\xrightarrow{\text{incl}_*}$ $S[-\infty, -\infty, p, b]$

$\dots$

$S[z, q, p, b]$



Short bibliography

R. Ghrist, *Barcodes: the persistent topology of data* (2008),

G. Carlsson, A. Zomorodian, *The theory of multidimensional persistence* (2007),

A. Romero, J. Heras, J. Rubio, F. Sergeraert, *Defining and computing persistent* $\mathbb{Z}$*-homology in the general case* (2014),

S. Basu, L. Parida, *Spectral sequences, exact couples and persistent homology of filtrations* (2015),

B. Matschke, *Successive spectral sequences* (2013),

A. G., F. Vaccarino, *Multidimensional persistence and spectral sequences*, in preparation.