An Isabelle/HOL Formalisation of Green's Theorem

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Abstract

- We formalised a statement of Green's theorem in Isabelle/HOL
- We avoid the traditional line integral cancellation argument
- Outline
  - What is Green's theorem?
  - Traditional statement and proof of Green's theorem
  - Glimpse of the new statement and proof

Stokes' Theorems

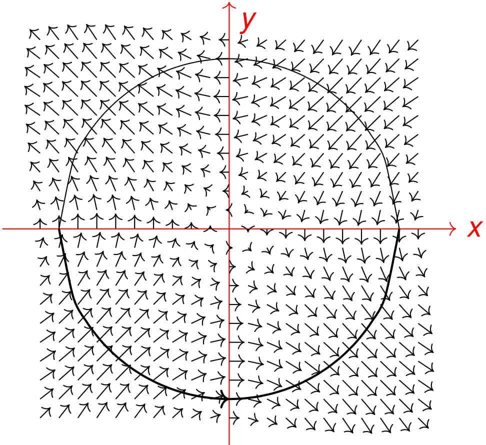
- A family of theorems relating functions to the integrals of their derivatives
- 1 dimension: Fundamental Theorem of Calculus, for $f : \mathbb{R} \Rightarrow \mathbb{R}$

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

- 2 dimensions: Green's Theorem for a field $F : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$ and a region in $\mathbb{R}^2$
- It was proven in 1828 by George Green

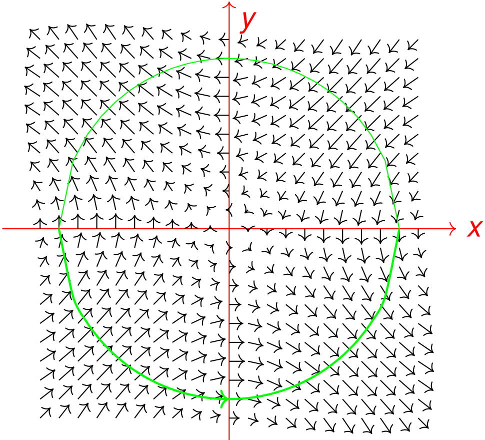
Green's Theorem

$$\oint_{\partial D} F_x dx + F_y dy = \int_{D} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$



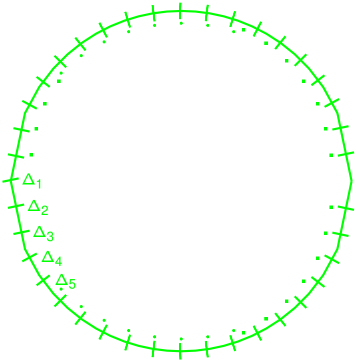
Line integral

$$\oint_{\partial D} F_x dx + F_y dy = \int_{D} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$



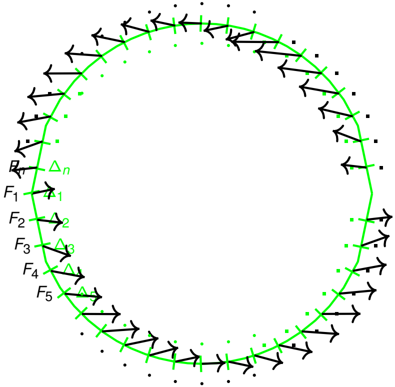
Line integral

$$\Delta_i = (x_{i+1} - x_i, y_{i+1} - y_i)$$



Line integral

$$\int_{\gamma} F \approx \sum_{1}^{n} F_i \bullet \Delta_i = \sum_{1}^{n} F_{x_i} \Delta_{x_i} + F_{y_i} \Delta_{y_i}$$



Line integral

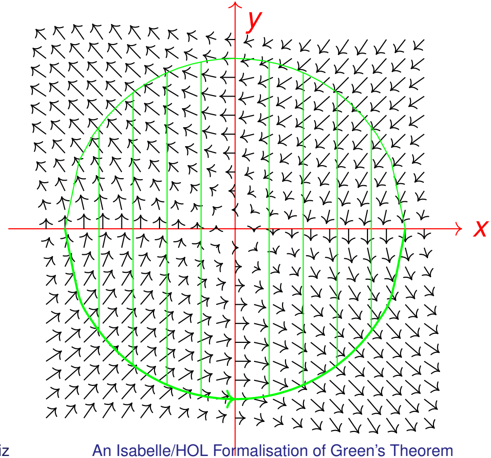
$$\int_{\gamma} F \approx \sum_{1}^{n} F_i \bullet \Delta_i = \sum_{1}^{n} F_{x_i} \Delta_{x_i} + F_{y_i} \Delta_{y_i}$$

This summation approximates:

- rotation of a field
- circulation of a fluid w.r.t. a boundary
- work done by a field on a particle

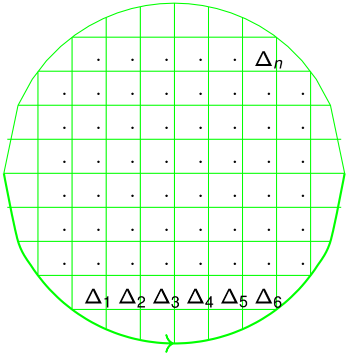
Double integral

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$



Double integral

$$\Delta_i = (x_{i+1} - x_i)(y_{i+1} - y_i)$$



Double integral

For a "scalar" function $g : \mathbb{R}^2 \Rightarrow \mathbb{R}$, the double integral can be approximated by the summation

$$\int_D g\, dxdy \approx \sum_{i=1}^{n} g(x_i, y_i)\Delta_i$$

In our case $g = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$, i.e.

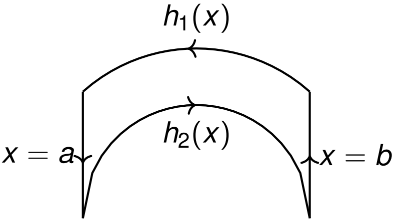
- the rate of change of the line integral w.r.t. area of D
- models the vorticity of a fluid, or field rotation density, etc.

Green's Theorem: Importance

- In mathematical analysis, e.g.
  - derive Cauchy's integral theorem
  - manipulating partial differential equations
- In analytical/mathematical physics, e.g.
  - electromagnetism and electrodynamics: e.g. deriving Faraday's law (point form)
  - astronomy: e.g. deriving Kepler's second law
- Justification of efficient numerical methods for
  - approximating integral on the boundary $O(n)$ vs $O(n^2)$
  - in fluid dynamics, image processing
- Justification of algorithms for hybrid systems

Green's Theorem: Type I Regions

$$\oint_{\partial D_x} F_x dx + F_y dy = \int_{D_x} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$

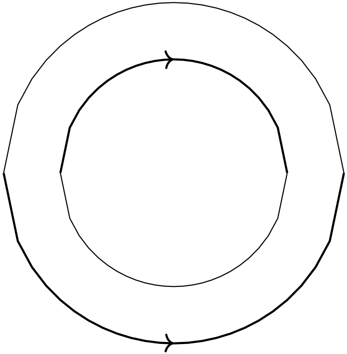


Proving theorem for type I region is relatively simple

- Line integral of F_x on y-axis is zero
- *Univariate change of variables*
- Fubini's theorem

Green's Theorem: Traditional Proof

For a region D that can be divided into a finite set C_x of Type I regions



Green's Theorem: Traditional Proof

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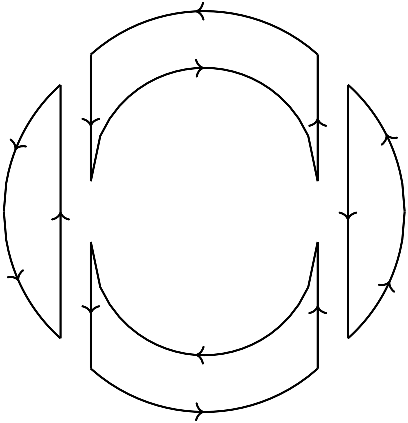


Green's Theorem: Traditional Proof

For a region D that can be divided into a finite set C_x of Type I regions

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$

Proof:

- $\oint_{\partial D} F_x dx = \sum_{D_x \in C_x} \oint_{\partial D_x} F_x dx$
- $\sum_{D_x \in C_x} \int_{D_x} - \frac{\partial F_x}{\partial y}\, dxdy = \int_D - \frac{\partial F_x}{\partial y}\, dxdy$
- Half Green's theorem for Type I regions

Green's Theorem: Traditional Proof

Difficulties of formalising this proof

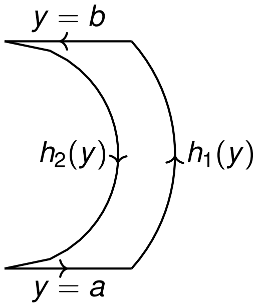
- complex/tedious topological argument of line integral cancellation
- formalising paths and their orientations w.r.t. exterior normal: complicated

We showed that line integral cancellation is not needed

- If D can be divided into type I regions C_x by adding *only* vertical lines
- Because the integral of F_x on any vertical line is zero

Green's Theorem: Type II Regions

$$\oint_{\partial D_y} F_x dx + F_y dy = \int_{D_y} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$



Green's Theorem: Traditional Proof

Similarly, for a region D that can be divided in finitely many Type II regions

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$

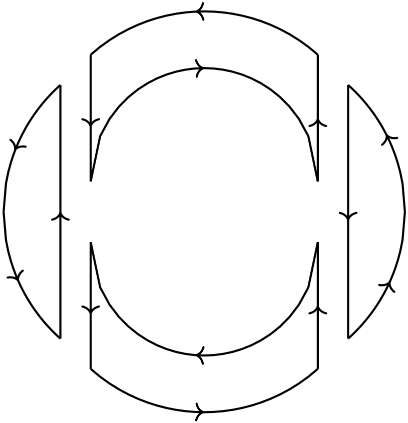
And accordingly we have:

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$

if D can be represented into *both* a set of type I regions *and* a set of type II regions.

Green's Theorem: Our approach

If *D* can be divided into type I regions C_x by adding *only* vertical lines



Green's Theorem: Our approach

If D can be divided into type I regions C_x by adding *only* vertical lines
We have

$$\int_{\gamma_x} F_x dx = \int_{D} -\frac{\partial F_x}{\partial y}\, dxdy$$

for any set of oriented paths γ_x that includes *all* the horizontal edges of C_x

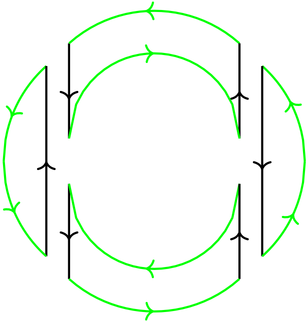
- Because the integral of F_x on any vertical line is zero

Green's Theorem: Our approach

If D can be divided into type I regions C_x by adding *only* vertical lines

$$\int_{\gamma_x} F_x dx = \int_D -\frac{\partial F_x}{\partial y}\, dxdy$$

γ_x can be

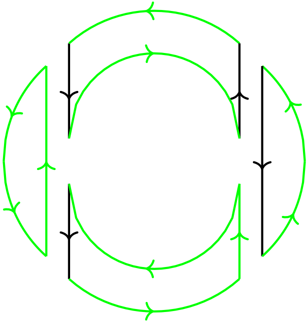


Green's Theorem: Our approach

If D can be divided into type I regions C_x by adding *only* vertical lines

$$\int_{\gamma_x} F_x dx = \int_D -\frac{\partial F_x}{\partial y}\, dxdy$$

γ_x can be

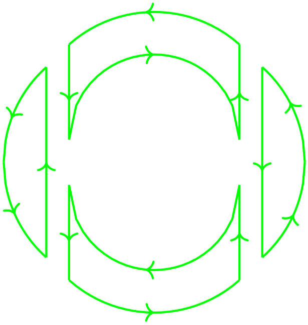


Green's Theorem: Our approach

If D can be divided into type I regions C_x by adding *only* vertical lines

$$\int_{\gamma_x} F_x dx = \int_D -\frac{\partial F_x}{\partial y}\, dxdy$$

γ_x can be



Green's Theorem: Our Approach: Path Equivalence

Similarly, if D can be divided into type II regions C_y *only* with horizontal lines

$$\int_{\gamma_y} F_y dy = \int_{D} \frac{\partial F_y}{\partial x} \, dxdy$$

- For any 1-chain γ_y that includes *all* the vertical boundaries of C_y
- Because the integral of F_y on any horizontal line is zero

Green's Theorem: Our Approach: Path Equivalence

We have

$$\int_{\gamma_x} F_x dy = \int_D -\frac{\partial F_x}{\partial y}\, dxdy$$

and

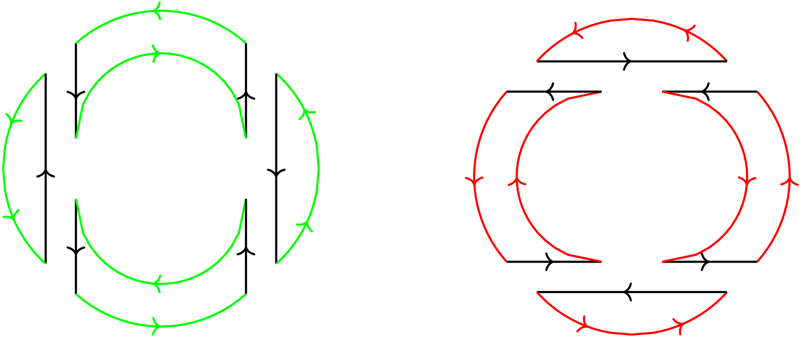
$$\int_{\gamma_y} F_y dy = \int_D \frac{\partial F_y}{\partial x}\, dxdy$$

How can we combine them?

- It is not straight-forward because γ_x and γ_y are not necessarily the same

Green's Theorem: Our Approach: Path Equivalence

Example γ_x, and γ_y



They are equivalent, but not the same

Green's Theorem: Our Approach: Path Equivalence

Their equivalence can be captured by

- formalising paths and their orientations w.r.t. exterior normal, OR
- the concept of a *common subdivision*

Green's Theorem: Our Approach: Path Equivalence

Reparameterisation:

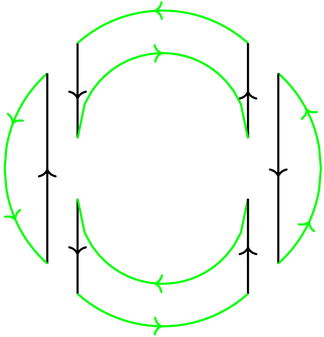
- γ_2 is a reparameterisation of γ_1 iff (roughly!)
  - $\exists \phi . \gamma_2 = \gamma_1 \circ \phi$

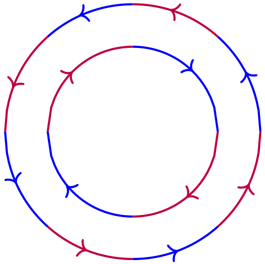
1-chain γ_1 is a subdivision of 1-chain γ_2 iff

- for every path (i.e. 1-cube) $c \in \gamma_2$,
  - there is a list of cubes from γ_1 that is a reparameterisation of c
- One way of capturing the equivalence of two 1-chains is the existence of a common subdivision

Green's Theorem: Our Approach: Path Equivalence

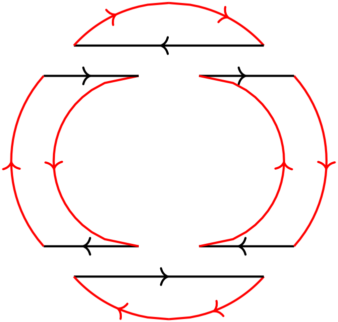
A subdivision of γ_x

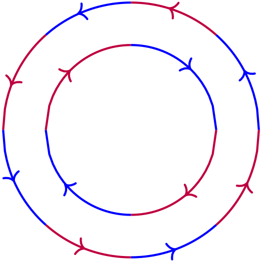




Green's Theorem: Our Approach: Path Equivalence

A subdivision of γ_y





Green's Theorem: Our Approach: Path Equivalence

1-chain γ_1 is a subdivision of 1-chain γ_2 iff

- for every cube $c \in \gamma_2$,
  - there is a list of cubes from γ_1 that subdivides c
- One way of capturing the equivalence of two 1-chains is the existence of a common subdivision

Lemma

For 1-chains γ_1 and γ_2, if there is a common subdivision between them, then

$$\int_{\gamma_1} F_x dx + F_y dy = \int_{\gamma_2} F_x d_x + F_y dy$$

Green's Theorem: Our Approach: Path Equivalence

Theorem (Green's Theorem)

If D can be represented by both a type I 2-chain C_x and a type II 2-chain C_y

- *using only vertical and horizontal lines, respectively.*

for any 1-chain γ_x that that includes all the horizontal edges of C_x

$$\oint_{\gamma_x} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$

Green's Theorem: Our Approach: Generality: Geometrical Assumptions

Conjecture

If D can be be represented both by type I 2-chain and type II 2-chain, then

- *D can be represented by both a type I 2-chain C_x and a type II 2-chain C_y*
  - *using only vertical and horizontal lines, respectively.*

Green's Theorem: Our Approach: Generality: Analytic Assumptions

Our theorem's analytic assumptions are

- F_x and F_y are continuous in D
- $\frac{\partial F_x}{\partial y}$ and $\frac{\partial F_x}{\partial y}$ exist and are Lebesgue integrable in D

More general than the assumption, commonly used in analysis books

- F and all of its partial derivatives are continuous in D

Green's Theorem: Discussion

- We conjecture that our statement is as general as the original one
- Previous formalisations that we used:
  - the Probability and the multivariate analysis libraries from Isabelle/HOL
  - Paulson's porting of Harrison's HOL light complex analysis
- Size of the formalisation 17K lines

Green's Theorem: Conclusions and Future Work

- We formalised a sufficiently general statement of Green's theorem
- This was facilitated by
  - a new argument that avoids line integral cancellation
  - a new formulation that avoid explicitly specifying the boundary w.r.t. the region
- As future work:
  - generalise this argument to prove the general Stokes' theorem
  - will at least need a multivariate change of variable theorem