Automatización de Análisis Asintótico en Isabelle/HOL

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Agenda

1. A Smörgåsbord of Asymptotic Analysis in Isabelle/HOL
2. Formal Aspects of Asymptotics in Isabelle
3. Automating Asympotics

Disclaimer

I did not invent any of these things.
I just figured out how to do them in Isabelle/HOL

A Smörgåsbord of Asymptotic Analysis in Isabelle/HOL

Euler–MacLaurin formula

Relates the value of a sum to the corresponding integral

$$\sum_{i=a+1}^{b} f(i) - \int_a^b f(x)\mathrm{d}x = \sum_{k=1}^{N} \frac{B_k}{k!} \left(f^{(k-1)}(b) - f^{(k-1)}(a) \right) + R$$

$$\text{where} \quad R = \frac{(-1)^N}{N!} \int_a^b \bar{B}_N(x) f(x)\mathrm{d}x$$

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Useful e. g. for $n!$, H_n, Γ, ψ, ...

Linear Recurrences

Theory and solver for linear recurrences with constant coefficients, i. e.

$$a_0 f(n) + \ldots + a_k f(n+k) = b_n \ .$$

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- Average-case analysis of algorithms (cf. Flajolet)
- Analysis of probabilistic programs: Random Walk

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$$f(n) = g(n) + \sum a_i f(b_i n + h_i(n))$$

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- Merge Sort: $f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + n, \implies \Theta(n \log n)$

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- Merge Sort: $f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + n, \implies \Theta(n \log n)$
- QuickSelect: $f(\lfloor \frac{1}{5} n \rfloor) + f(\lfloor \frac{7}{10} n \rfloor + 6) + \frac{12}{5} n, \implies \Theta(n)$

Analytic Combinatorics

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- Via Gamma function: $C_n \sim \frac{4^n}{\sqrt{\pi} n^{1.5}}$

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Can also be used to prove $\zeta(2n) = \frac{1}{2}(-1)^{n+1} B_{2n} \frac{(2\pi)^{2n}}{(2n)!}$.

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- Akra–Bazzi: Merge Sort, Karatsuba, QuickSelect
- Linear Recurrences: Expected length of a Random Walk
- Dirichlet series: Expected number of divisors; density of squarefree numbers, coprime numbers
- Other: General comparison sorting, QuickSort, BSTs, Treaps, Skip Lists

Formal Aspects of Asymptotics in Isabelle

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$$\lim_{x\to\infty} f(x) = c \quad \hat{=} \quad \text{filterlim } f \ (\text{nhds } c) \text{ at\_top}$$

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' $f(x)$ goes to F as as $x \to G$ iff neighbourhood G mapped with f is contained in neighbourhood F. '

Filters also make it possible to say '$P(x)$ holds for all x sufficiently big / close to x_0.'

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What does 'QuickSort needs $O(|xs|^2)$ comparisons' mean?

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$$\text{qs\_cost} \in O[\text{length going\_to at\_top}](\lambda\text{xs}.\ (\text{length } xs)^2)$$

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$$(\lambda x.\ x^a * (\ln x)^b) \in O(\lambda x.\ x^c * (\ln x)^d) \quad \rightsquigarrow$$
$$a < c \lor (a = c \land b \leq d)$$

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$$(\lambda x.\ x^a * (\ln x)^b) \in O(\lambda x.\ x^c * (\ln x)^d) \quad \rightsquigarrow \quad a < c \vee (a = c \wedge b \leq d)$$

Things like $(\lambda x.\ 2 * x + x * \ln \ln x) \in O(\lambda x.\ x * \ln x)$ get proven automatically.

Automating Asymptotics

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Example: Lemma required for Akra–Bazzi

$$\lim_{x\to\infty} \left(1 - \frac{1}{b\log^{1+\varepsilon} x}\right)^p \left(1 + \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) - \left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) \;=\; 0^+$$

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In Isabelle: 700 lines of messy proofs

lemma akra_bazzi_aux:
filterlim
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$(1 + \ln\ (b * x + x/\ln x\ \hat{}\ (1+\varepsilon))\ \hat{}\ (-\varepsilon/2)) -$
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This is what we would like to have.

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This is what we would like to have.

Computer Algebra Systems can do this (sort of)

So why can't we?

Asymptotic Expansions

Related Work

- *Asymptotic Expansions of exp–log Functions* by Richardson, Salvy, Shackell, van der Hoeven
- *On Computing Limits in a Symbolic Manipulation System* by Gruntz

Asymptotic Expansions

For $x \to \infty$, we have:

$$e^{1/x} \sim 1 + x^{-1} + \tfrac{1}{2}x^{-2} + \tfrac{1}{6}x^{-3} + \dots$$

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This means: Cutting off $f(x) \sim a_0(x) + a_1(x) + \dots$ at term a_n yields error $O(a_{n+1}(x))$.

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Expansions contain the *full* asymptotic information.

They can be added/subtracted/multiplied/divided.

Limits can simply be 'read off'

Asymptotic Expansions

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For now, we only consider expansions of the form

$$f(x) \sim c_0 x^{e_0} + c_1 x^{e_1} + \dots$$

for $x \to \infty$ where $e_0 > e_1 > \dots$

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&(+) : \mathsf{Exp} \to \mathsf{Exp} \to \mathsf{Exp} \\
&[] \ + \ ys \ = \ ys \\
&xs \ + \ [] \ = \ xs \\
&((c_1, e_1) :: xs) \ + \ ((c_2, e_2) :: ys) \\
&\quad \mid e_1 == e_2 \ = \ (c_1 + c_2, e_1) \ :: \ xs + ys \\
&\quad \mid e_1 < e_2 \ = \ (c_1, e_1) \ :: \ xs + ((c_2, e_2) :: ys) \\
&\quad \mid e_1 > e_2 \ = \ (c_2, e_2) \ :: \ ((c_1, e_1) :: xs) + ys
\end{aligned}$$

Asymptotic Expansions – Multiplication

Multiplication with 'atomic' factor $c'x^{e'}$:

$$\text{scale} \,:\, \mathbb{R} \to \mathbb{R} \to \text{Exp} \to \text{Exp}$$
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$$[] * ys \ = \ []$$
$$((c_1, e_1) :: xs) * ((c_2, e_2) :: ys) \ =$$
$$(c_1 * c_2, e_1 + e_2) \ :: \ \text{scale}\ c_1\ e_1\ ys + xs * ((c_2, e_2) :: ys)$$

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Example: $(e^x, x, \ln x)$ is an asymptotic basis and generates monomials $e^{ax} x^b \ln^c x$

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- Lots of opportunities for implementation bugs
- Luckily, the Isabelle kernel caught them, of course. :)

Proof method

Skipping a lot of magic: We can automatically prove statements of the form

- $f(x) \longrightarrow c$
- $f(x) \sim g(x)$
- $f(x) \in L(g(x))$ for any Landau symbol L

as $x \to l$ for $l \in \mathbb{R} \cup \{\pm\infty\}$

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sin, cos, tan at finite points also possible.

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lemma $((\lambda x.\ (1 + y/x)\ \hat{}\ x) \longrightarrow \mathrm{exp}\ \ y)$ at_top
proof (cases $y = 0$)
  case False
  thus ?thesis **by** exp_log_asymptotics
qed simp_all

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lemma
assumes $c > 1$ **and** $k > 0$
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  by (exp_log_asymptotics simp: mult_neg_pos)

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- Getting zeroness/sign tests to work can be annoying
- Case distinctions have to be done manually

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- Implementation was tricky to get right

Comparison to CASs

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Maxima and Sage take very long for some of them
and give wrong result for this:

$$\exp\left(\frac{\log\log\left(x + e^{\log x \log\log x}\right)}{\log\log\log\left(e^x + x + \ln x\right)}\right) \longrightarrow e$$

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Isabelle still isn't a CAS – but we're getting there.

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- User interaction for zeroness tests could be improved

Questions? Demo?